Fun with Binomial Coefficients

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Oberlin College
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Outline

Binomial comes from the Latin

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The *binomial coefficients* are the coefficients of $1 + x$ raised to various powers.

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so $(1+x)^3$ has binomial coefficients $1, 3, 3, 1$.

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & + & x & & & \\
 & & 1 & + & 2x & + & x^2 & & \\
 & 1 & + & 3x & + & 3x^2 & + & x^3 & \\
1 & + & 4x & + & 6x^2 & + & 4x^3 & + & x^4 \\
 & & & & \vdots & & & &
\end{array}
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 & & & & \vdots & & & &
\end{array}
$$

Writing down just the coefficients gives *Pascal's Triangle*

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
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\end{array}
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Theorem

We have the boundary conditions for the values $k = 0$ and $k = n$

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In Pascal's Triangle the recursion translates as

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				1				
			1		1			
		1		2		1		
	1		3	+	3		1	
1		4		6				1

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Combinatorics is the area of mathematics which deals with counting discrete structures.

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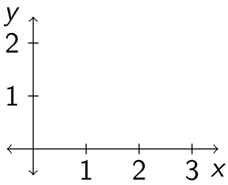
$P =$ 

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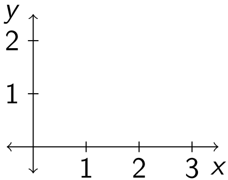
$P =$ 

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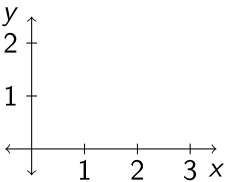
$P =$ 

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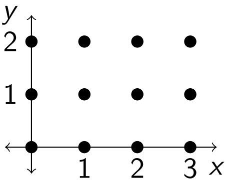
The *integer lattice* is

$$\mathbb{Z}^2 = \{(x, y) \text{ in } P \text{ such that both } x, y \text{ are integers}\}.$$

$\mathbb{Z}^2 =$ 

A *lattice path* to the point (x, y) in $\mathbb{Z}^2$

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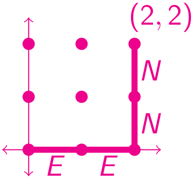

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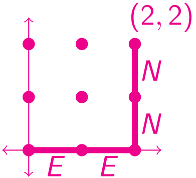

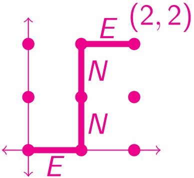

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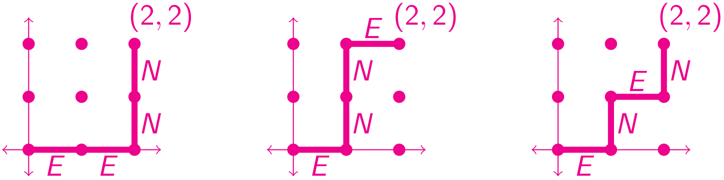

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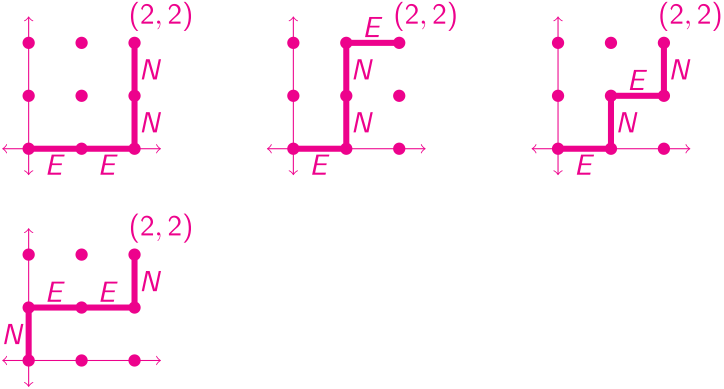

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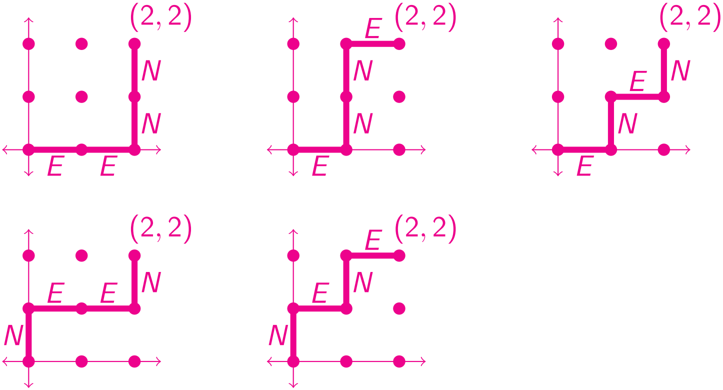

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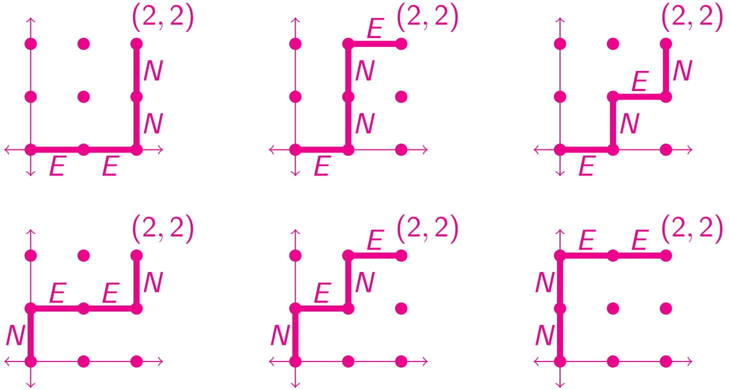

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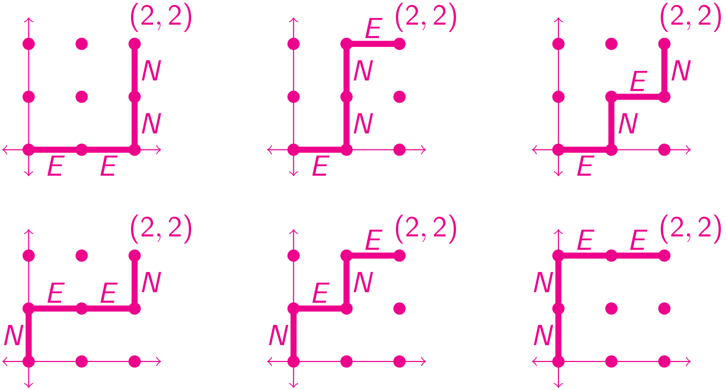

So 6 possible paths.

Here is a table where lattice points (x, y) have been labeled by the number of lattice paths to (x, y)

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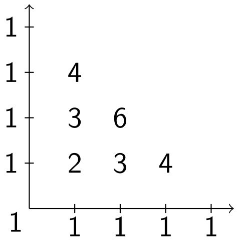

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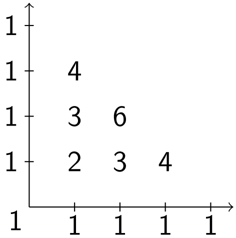

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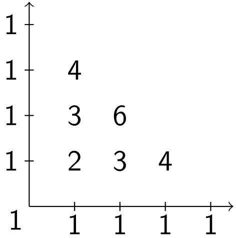

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Theorem

The number of lattice paths to (x, y) is $\binom{x+y}{x}$.

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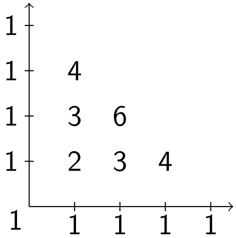

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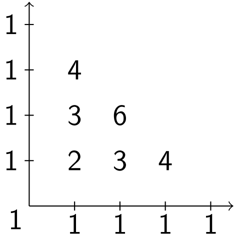

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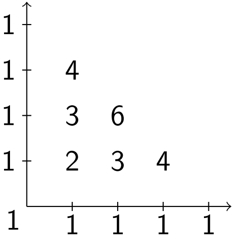

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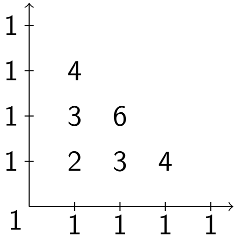

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The number of lattice paths to (x, y) is $\binom{x+y}{x}$.

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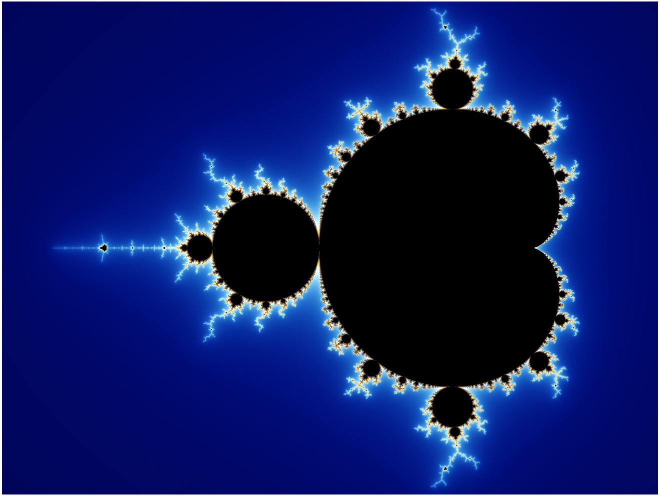
$$\text{number of lattice paths to } (2,2) = \binom{2+2}{2} = \binom{4}{2} = 6.$$

Outline

A mathematical object is *fractal* if it displays the same characteristics at different scales.

A mathematical object is *fractal* if it displays the same characteristics at different scales. The *Mandelbrot set* is an example



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$$
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 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
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1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

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							1							
						1		1						
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				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
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				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
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							1							
						1		1						
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1 1

1 0 1

1 1 1 1

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Consider the triangle in the first two rows and compare with the next two rows.

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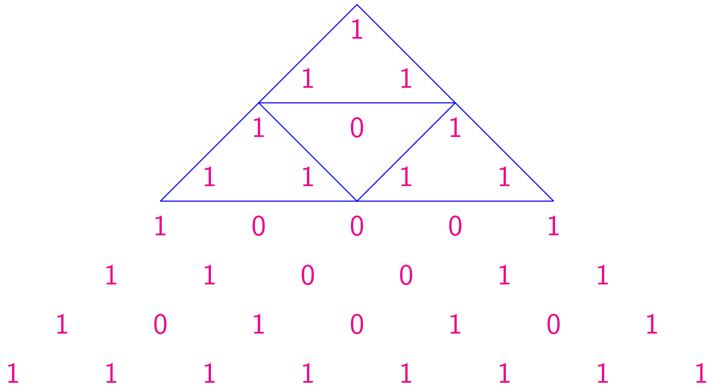

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 & & & & & & 1 & & 1 & & & & & & \\
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 & & & 1 & & 0 & & 0 & & 0 & & 1 & & & \\
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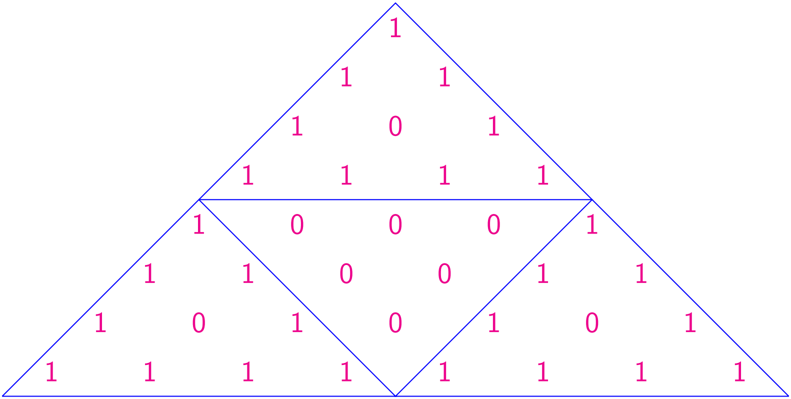

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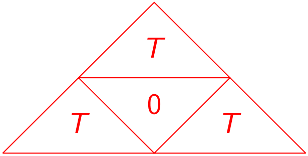
Theorem

If the first 2^n rows of Pascal's Triangle form a triangle T then the first 2^{n+1} rows of Pascal's triangle are



Where the central triangle is all zeros.

Outline

The *Fibonacci numbers* are $F_1 = 1$, $F_2 = 1$, and for $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}.$$

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Ex.

n	1	2	3	4	5	6
F_n						

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Ex.

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F_n	1	1	2	3		

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Ex.

n	1	2	3	4	5	6
F_n	1	1	2	3	5	8

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The *nth fibatorial* is

$$F_n^! = F_1 \cdot F_2 \cdot F_3 \cdots F_n.$$

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Ex.

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Ex. $F_5^!$.

The *Fibonacci numbers* are $F_1 = 1$, $F_2 = 1$, and for $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}.$$

Ex.

n	1	2	3	4	5	6
F_n	1	1	2	3	5	8

The *nth fibatorial* is

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Ex. $F_5^! = F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot F_5$.

The *Fibonacci numbers* are $F_1 = 1$, $F_2 = 1$, and for $n \geq 2$

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The *Fibonacci numbers* are $F_1 = 1$, $F_2 = 1$, and for $n \geq 2$

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These numbers are always positive integers although this is **not** clear from the definition.

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3. *(Chen and S) The fibonomial triangle modulo* 2 *is fractal using triangles of size* $3 \cdot 2^n$*.*

Note that the first property makes it easy to prove that the fibonomials are always integers using mathematical induction.

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Open Question

What is the period of the Fibonacci sequence modulo m?

Outline

1. Amdeberhan, Tewodros; Chen, Xi; Moll, Victor H.; Sagan, Bruce E. Generalized Fibonacci polynomials and Fibonomial coefficients. Ann. Comb. 18 (2014), no. 4, 541–562.

2. Bennett, Curtis; Carrillo, Juan; Machacek, John; Sagan, Bruce E. Combinatorial interpretations of Lucas analogues of binomial coefficients and Catalan numbers. Ann. Comb. 24 (2020), no. 3, 503–530.

3. Chen, Xi; Sagan, Bruce E. The fractal nature of the Fibonomial triangle. Integers 14 (2014), Paper No. A3, 12 pp.

4. Sagan, Bruce E. Combinatorics: the art of counting. Graduate Studies in Mathematics, 210. American Mathematical Society, Providence, RI, [2020], ©2020. xix+304 pp. ISBN: 978-1-4704-6032-7

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THANKS FOR LISTENING!